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THE UNIVERSITY OF TENNESSEE

DEPARTMENT OF ELECTRICAL ENGINEERING

DESIGN OF TIME WEIGHTED MINIMUM ENERGY
DISCRETE-DATA CONTROL SYSTEMS

by

A. M. Revington and J. C. Hung

Supported by National Aeronautics and Space Administration
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SUMMARY

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A procedure is developed for designing the time weighted minimal energy control of an n-th order plant. It is desired to take the state of the plant from some initial state to some required final state in N sampling periods, with $N > n$, while minimizing a time weighted cost function of the controlling energy. This cost function stresses the controlling action of any part of the input sequence and relegates the remainder of the sequence to a supporting role.

The control sequence is found in a very straightforward manner, depending only on a very useful matrix called the derived matrix. ~~A~~ detailed example shows how this matrix is utilized and demonstrates the advantages of this time weighted minimal energy control sequence.

AUTHOR

INTRODUCTION

The discrete regulator is assuming an increasingly important place in modern control systems. Discrete regulators are known by the type of input modulation; pulse amplitude modulated signals followed by a zero order hold (abbreviated to PAM in this report) implies that the input is held at a constant level during a sampling period, and allowed changes in level only at the sampling instants. The state of an n-th order linear time invariant PAM plant is described by

$$\underline{x}\left[(k+1)T\right] = G(T)\underline{x}(kT) + \underline{h}(T)\, m\left[(k+1)T\right]$$

where $\underline{x}\left[(k+1)T\right]$ is an n-vector, the state of the system at the k+1-th sampling instant. $G(T)$ is an nxn matrix called the state transition matrix and $\underline{h}(T)$ is an nx1 matrix called the forcing matrix. $m\left[(k+1)T\right]$ is the input level over the time interval $kT \leq t < (k+1)T$. The matrices $G(T)$ and $\underline{h}(T)$ are obtained from the plant transfer function by standard techniques[1].

Kalman and Bertram[2,3] defined and then developed the 'canonical' vectors as the basic building blocks of the PAM regulator. These vectors, $\underline{r}_i$, $i = 1, 2, \ldots .N$, $N > n$ are defined in terms of the state transition and forcing matrices as

$$\underline{r}_i = G(T)^{-i}\,\underline{h}(T) = G(-iT)\underline{h}(T) \qquad i = 1,2, \ldots .n, \ldots .N.$$

The reader unfamiliar with their use is recommended to read the papers[2] which first introduced and then made excellent use of them. They can make a seemingly difficult sampled-data problem seem simple. We shall show that by a further extension of this concept, namely the formation and employment of the "derived matrix," an even more difficult problem, that of energy minimization, can be reduced to a set of surprisingly simple formulae.

In the development of the application of the canonical vector concept it was shown[3] that any initial state $\underline{x}$ could be represented by a linear combination of these canonical vectors:

$$\underline{x} = \sum_{k=1}^{N} a_k\,\underline{r}_k \quad , N > n$$

where the a_k are scalars. It was also shown that this state could be taken to the origin of the state space by applying the PAM input sequence

$$m(kT) = -a_k \ , \quad k = 1,\ 2,\ \ldots .\ N. \tag{1}$$

Thus was demonstrated an elegant mathematical conception of the discrete regulator which laid the foundations for further work. The regulator problem was essentially reduced to finding a representation of the disturbed state $\underline{x}$ as a linear combination of the canonical vectors.

In this paper we shall derive formulae that can be directly applied to the problem of taking the disturbed state $\underline{x}$ to the origin of the state space while minimizing a quadratic cost function of the members of the input sequence. In a previous report[4] techniques were developed to handle the so-called minimal energy problem. The more general cost function considered here may be regarded as the time weighted extension of the previous report.

MATHEMATICAL DEVELOPMENT

Problem Statement

Given the linear n-th order time invariant PAM plant described by

$$\underline{x}\left[(k+1)T\right] = G(T)\underline{x}(kT) + \underline{h}(T)\ m\left[(k+1)T\right] \tag{2}$$

it is desired to find the input sequence $m(kT)$, $k = 1, 2, \ldots .N$ that will take any given initial state $\underline{x}$ to be the origin of the state space X in N sampling periods, and that will also minimize the cost function

$$E = \sum_{k=1}^{N} d(k)\ m(kT)^2 \tag{3}$$

where the $d(k)$ are positive scalars. When $d(k) = 1$ for all k this problem reduces to the minimal energy problem[4].

Theory

Associated with the system of Eq. 2 we have the N canonical vectors, N an integer, $N > n$.,

$$\underline{r}_k = G\ (-kT)\ \underline{h}(T) \qquad k = 1, 2, \ldots .\ N. \tag{4}$$

Thus a general representation of the initial disturbed state is

$$\underline{x} = \sum_{k=1}^{N} a_k \underline{r}_k\ . \tag{5}$$

For "completely controllable" systems[5] n of the $\underline{r}_k$ are linearly independent. Now $\underline{x}$ is an n-vector, so that for $N > n$ the representation of $\underline{x}$ in Eq. 5 is not unique, in other words there are an infinite number of representations, input sequences, which can be used to take $\underline{x}$ to the origin in $N > n$ sampling periods. If $N = n$ there is a unique solution; the solution to the set of n simultaneous equations

$$\underline{x} = \sum_{k=1}^{n} a_k \underline{r}_k$$

is the unique solution to the linear time-optimal regulator problem. We assume however $N > n$, so that the additional problem of minimizing the cost function, Eq. 3, has meaning.

Rewriting the representation of Eq. 5,

$$\underline{x} = \sum_{k=1}^{n} a_k \underline{r}_k + \sum_{k=1}^{N-n} b_k \underline{r}_{n+k} \tag{6}$$

giving

$$\underline{x} = R\underline{a} + Q\underline{b} \tag{7}$$

where R is the nxn matrix, the columns of which are the n canonical vectors $\underline{r}_k$, k=1,2,..n. Q is the nx(N-n) matrix with columns the canonical vectors $\underline{r}_{n+1}, \underline{r}_{n+2}, \ldots \underline{r}_N$. $\underline{a}$ is the n-vector $(a_1, a_2, \ldots a_n)^t$ and $\underline{b}$ the (N-n) vector $(b_1, b_2, \ldots b_{N-n})^t = (a_{n+1}, a_{n+2}, \ldots a_N)^t$, where "t" denotes the transpose of a matrix. The components of $\underline{a}$ and $\underline{b}$ represent, except for a change in sign, the input sequence to be applied to the plant. From Eq. 7,

$$R^{-1}\underline{x} = \underline{a} + R^{-1}Q\underline{b} \ . \tag{8}$$

Define

$$\begin{aligned} \underline{c} &= R^{-1}\underline{x} \\ H &= R^{-1}Q \ . \end{aligned} \tag{9}$$

This premultiplication of Eq. 7 by the matrix R^{-1} effectively takes the vector $\underline{x}$ in X, the state space, to the vector $\underline{c}$ in C, where C is a new n-space, the coordinates of which are the first n canonical vectors $\underline{r}_1, \underline{r}_2, \ldots \underline{r}_n$. The reason for doing this is merely to simplify later manipulations. We shall call C the "canonical vector space". $\underline{c}$ is merely the initial state $\underline{x}$ expressed as a linear combination of the first n canonical vectors. The nx(N-n) matrix H, which we shall call the "derived matrix", having derived it from the last N-n canonical vectors, is as fundamental to the regulator problem with more than n inputs as the matrix R^{-1} is fundamental to the simple linear time optimal regulator. Thus by premultiplying the original representation, Eq. 6, by R^{-1} we have the considerably simplified representation,

$$\underline{c} = \underline{a} + H\underline{b} \ . \tag{10}$$

The energy consumption can also be written compactly as

$$E = \underline{a}^t D\underline{a} + \underline{b}^t F\underline{b} \tag{11}$$

where D is the diagonal matrix with elements $d_{kk} = d(k)$, k=1,2,..n, and F is the (N-n)x(N-n) diagonal matrix with elements $f_{kk} = d(k+n)$, k=1,2,...N-n. From Eq. 10,

$$\underline{a} = \underline{c} - H\underline{b} \ . \tag{12}$$

Substituting $\underline{a}$ from Eq. 12 into Eq. 11 reduces the energy function to a function of N-n independent variables, the components of $\underline{b}$, giving

$$E = (\underline{c} - H\underline{b})^t \; D \; (\underline{c} - H\underline{b}) + \underline{b}^t \; F \; \underline{b} \tag{13}$$

$$E = (\underline{c}^t - \underline{b}^t H^t) \; D \; (\underline{c} - H\underline{b}) + \underline{b}^t \; F \; \underline{b} \tag{14}$$

$$E = \underline{c}^t D\underline{c} - 2\underline{c}^t DH\underline{b} + \underline{b}^t \left[H^t DH + F \right] b \tag{15}$$

Differentiating E with respect to each of the components of $\underline{b}, b_p$, $p = 1, 2, \ldots\ldots N\text{-}n$, we find (see Appendix I) by setting $\frac{\partial E}{\partial b_p} = 0$ that a minimum of E is obtained when

$$\left[H^t \; DH + F \right] \underline{b} = H^t \; D\underline{c}. \tag{16}$$

Substitute Eq. 10 into Eq. 16, eliminating $\underline{c}$, we obtain

$$H^t \; DH\underline{b} + F\underline{b} = H^t D\underline{a} + H^t CH\underline{b}. \tag{17}$$

Then

$$F\underline{b} = H^t D\underline{a} \tag{18}$$

Substitute Eq. 18 into Eq. 11, to obtain

$$\underline{a} = \underline{c} - HF^{-1} H^t \; D\underline{a} \tag{19}$$

so that we have

$$\left[I + HF^{-1} H^t \; D \right] \underline{a} = \underline{c} \tag{20}$$

where I is the nxn identify matrix.

Let

$$B = I + HF^{-1} H^t \; D \tag{21}$$

Then

$$\underline{a} = B^{-1} \underline{c} \tag{22}$$

The existence of B^{-1} is proved in Appendix II. Now we obtain the final equation giving a simple expression for the value of the minimal energy by substituting Eq. 18 into Eq. 11 giving

$$E = \underline{a}^t D\underline{a} + \underline{b}^t H^t D\underline{a} = \left[\underline{a}^t + \underline{b}^t \; H^t \right] D\underline{a} \tag{23}$$

Thus

$$E = \underline{c}^t D \underline{a} \tag{24}$$

In summary, the important equations are repeated below.

From the representation of the initial state $\underline{c}$ in C we had

$$\underline{c} = \underline{a} + H\,\underline{b} \tag{25}$$

The energy consumption in taking $\underline{c}$ to the origin is then

$$E = \underline{a}^t D \underline{a} + \underline{b}^t F \underline{b} \tag{26}$$

which is minimized when $F\underline{b} = H^t\ D\underline{a}$, with the minimum value $E = \underline{c}^t\ D\underline{a}$ where

$$\underline{a} = B^{-1}\underline{c} \tag{27}$$

and

$$\underline{b} = F^{-1}\ H^t\ D\underline{a}. \tag{28}$$

Eq. 27 and Eq. 28 give with Eq. 1

$$\left[m(1),\ m(2),\ \ldots.\ m(n)\right] = \left[-a_1,\ -a_2,\ \ldots.\ -a_n\right] = -\underline{a}^t$$

$$\left[m(n+1),\ m(n+2),\ \ldots.\ m(N)\right] = \left[-b_1,\ -b_2,\ \ldots.\ -b_{N-n}\right] = -\underline{b}^t$$

To demonstrate the way in which these results are applied to a practical problem we illustrate a suggested design procedure by a worked example.

EXAMPLE

Consider the simple second order plant described by the transfer function below, whose input is the output of a zero order hold, M(s), and whose output is Y(s). Then

$$\frac{Y(s)}{M(s)} = \frac{1}{s(s+1)} \tag{29}$$

Let the state variables be $x_1(t) = y(t)$, $x_2(t) = \dot{y}(t)$. $y(t)$ is the time dependent system output, and $\dot{y}$ the time derivative of the system output (position and velocity for example). Eq. 29 then gives the following vector differential equation,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} m \tag{30}$$

where m is the system input. For a given initial condition $\underline{x}(0)$, the solution of Eq. 30 is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 1-e^{-t} \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x^2(0) \end{bmatrix} + \begin{bmatrix} e^{-t} + t-1 \\ 1 - e^{-t} \end{bmatrix} m(t) \tag{31}$$

Eq. 31 may be expressed in compact form as

$$\underline{x}(t) = G(t)\underline{x}(0) + \underline{h}(t)\, \underline{m}(t) \tag{32}$$

Then in the system representation of Eq. 2,

$$G(T) = \begin{bmatrix} 1 & 1-e^{-T} \\ 0 & e^{-T} \end{bmatrix} \quad \text{and} \quad \underline{h}(T) = \begin{bmatrix} e^{-T}+T-1 \\ 1-e^{-T} \end{bmatrix} \tag{33}$$

With a sampling period T = 1 sec, the canonical vectors are given by Eq. 4, for k=1, 2, ... N

$$\underline{r}_k = \begin{bmatrix} -(e^k - e^{k-1} - 1) \\ e^k - e^{k-1} \end{bmatrix} \tag{34}$$

Let us arbitrarily choose N=4. Then

$$R = \begin{bmatrix} -0.7182 & -3.6706 \\ 1.7182 & 4.6706 \end{bmatrix} \tag{35}$$

and

$$Q = \begin{bmatrix} -11.6961 & -33.5118 \\ 12.6961 & 34.5118 \end{bmatrix} \tag{36}$$

Then we obtain the derived matrix $H = R^{-1}Q$ as

$$H = \begin{bmatrix} -2.7183 & -10.107 \\ 3.7183 & 11.107 \end{bmatrix} \tag{37}$$

Let us also choose the initial state $x_1(0) = 1$, $x_2(0) = 0$. Then

$$\underline{c} = R^{-1}\,\underline{x}(0) = \begin{bmatrix} 1.582 \\ -0.582 \end{bmatrix} \tag{38}$$

We readily find $I + H\,F^{-1}H^t\,D$ is given by

$$\begin{bmatrix} 1+d_{11}\left[\dfrac{2.7183^2}{f_{11}} + \dfrac{10.107^2}{f_{22}}\right] & d_{22}\left[\dfrac{-2.7183\text{x}3.7183}{f_{11}} - \dfrac{10.107\text{x}11.107}{f_{22}}\right] \\ d_{11}\left[\dfrac{-2.7183\text{x}3.7183}{f_{11}} \dfrac{-10.107\text{x}11.107}{f_{22}}\right] & 1+d_{22}\left[\dfrac{3.7183^2}{f_{11}} + \dfrac{11.107^2}{f_{22}}\right] \end{bmatrix} \tag{39}$$

Let us consider two cases.

(a) No time weighting $D = I$, $F = I$

(b) Time weighting, $D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ $F = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$

Case(a) $D = I$, $F = I$

From Eq. 27, $\underline{a} = B^{-1}\underline{c}$, we obtain

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0.457 & 0.405 \\ 0.405 & 0.366 \end{bmatrix} \begin{bmatrix} 1.582 \\ -0.582 \end{bmatrix} = \begin{bmatrix} 0.4875 \\ 0.4275 \end{bmatrix} \tag{40}$$

and from Eq. 28, $\underline{b} = F^{-1}\,H^t\,D\underline{a} = H^t\,\underline{a}$

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} -2.7183 & 3.7183 \\ -10.107 & 11.107 \end{bmatrix} \begin{bmatrix} 0.4875 \\ 0.4275 \end{bmatrix} = \begin{bmatrix} 0.2643 \\ -0.1794 \end{bmatrix} \tag{41}$$

Case (b). With $d_{11} = 1$, $d_{22} = 2$, $f_{11} = 3$, $f_{22} = 4$, we obtain from Eq. 39 and Eq. 27,

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0.6596 & 0.5768 \\ 0.2884 & 0.2661 \end{bmatrix} \begin{bmatrix} 1.582 \\ -0.582 \end{bmatrix} = \begin{bmatrix} 0.7078 \\ 0.3014 \end{bmatrix} \tag{42}$$

and from Eq. 28

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/4 \end{bmatrix} \begin{bmatrix} -2.7183 & \frac{3.7183}{2} \\ -10.107 & \frac{11.107}{2} \end{bmatrix} \begin{bmatrix} 0.7078 \\ 0.3014 \end{bmatrix} = \begin{bmatrix} 0.1056 \\ -0.1149 \end{bmatrix} \tag{43}$$

Thus we have found the minimal energy representation (case a) and the time weighted minimal energy representation (case b) for the initial state $x_1 = 1$, $x_2 = 0$. It is interesting to compare the input energies of the different cases.

We find,

Case (a) $\quad a_1^2 + a_2^2 + b_1^2 + b_2^2 = 0.5226, \quad N = 4$

Case (b) $\quad a_1^2 + a_2^2 + b_1^2 + b_2^2 = 0.6161, \quad N = 4$

Case (c) $\quad a_1^2 + a_2^2 = 2.8415, \quad N = 2$ (time-optimal)

We see that although the time-optimal control takes only two sampling periods to settle, giving the fastest response when saturation does not limit the input, it consumes over five times the energy of the minimal energy case and more than four times that of the time weighted energy case.

The representation, the vectors $\underline{a}$ and $\underline{b}$, can be used in two ways.

1. From Eq. 1, if we allow

$$m(k) = -a_k, \; k=1,2$$

$$m(k+2) = -b_k, \; k=1,2$$

as the input sequence then we can consider the initial state to be some disturbed state that we wish to return to the desired state, the origin of the state space.

2. If we allow

$$m(k) = + a_k, \; k=1, \; 2$$

$$m(k+2) = +b_k, \; k=1, \; 2.$$

to be the input sequence then the "initial state" $\underline{x}$ can be considered as the desired final state and the origin the initial state. This sequence will be used in the calculations following.

By a suitable transformation the origin of the state space can easily be moved to any desired point, and we can then use the vectors $\underline{a}$ and $\underline{b}$ as the input sequence to take any state to or from any other state, in N sampling periods and with a time weighted energy constraint.

A plot of the output of the plant with time is given in Fig. 1, for the two different cost functions. A third plot, that of the time-optimal case[2] is also shown for comparison. The inputs required for each output are shown in Fig. 2. The slope of the output, the velocity, can be found at the time when the outputs are at 90% of their final value. These velocities are, see Fig. 1,

$$\begin{aligned} &\text{Case (a)} \quad 0.262 \text{ at } t \approx 3.1 \text{ secs.} \\ &\text{Case (b)} \quad 0.206 \text{ at } t \approx 2.9 \text{ secs.} \\ &\text{Case (c)} \quad 0.377 \text{ at } t \approx 1.5 \text{ secs.} \quad \text{(time-optimal)} \end{aligned} \tag{44}$$

DISCUSSION

In the light of the example the advantages of the method are clearly seen. The method allows the immediate and straightforward calculation of an input sequence which not only considerably reduces the chances of saturation but also, when D and F are both identity matrices, minimizes the energy required by the system. The time weighted case can be used when, for some practical reason, it is desired to place the main burden of control on a certain section of the input sequence. In the example we emphasized the role of the early inputs and lessened the effect of the later inputs. This control still retains the deadbeat response and the ease of calculation of the inputs, while still requiring much smaller input magnitudes and input energy than the time optimal regulator. Of considerable value, however, is the fact that we approach the final state gradually, as shown in Eq. 44. A descending vehicle should not have high velocities near the touchdown point, as any errors in measurement could not be corrected in time. In general, by saving several sampling periods for the final settling time we should have enough time to allow for any error that might exist. We also observe, from Fig. 1, that the time weighted minimum energy control gives a faster rise time the the ordinary minimum energy control.

In summary, the advantages of the time weighted minimum energy control over minimum energy control are that by a suitable choice of D and F, it is possible to approach the terminal state more gradually and yet, for a given $N > n$, have a faster rise time. It should be noted that these improvements are not restricted to the particular D and F used in the example, so that other choices could give still greater improvements.

CONCLUSIONS

A procedure has been developed for designing the time weighted minimum energy discrete-data control of an n-th order plant, extending the results of a previous report [4]. The advantages of the design have been demonstrated and discussed with the aid of an example. The basis of the design procedure is the derived matrix, the use of which gives the desired control in a very simple manner. The procedure can be used with plants of any order and type.

It has been demonstrated that the chances of saturation are reduced by increasing the allowed settling time, N sampling periods. Since in many cases time optimal control is not of primary importance this method can be sucessfully applied to the problem of input saturation.

A forthcoming report will discuss the relationships between the number of sampling periods and the corresponding minimum energy for a given initial state. Such relationships can be developed with the use of the derived matrix and can be employed by the system designer to choose, for example, the value of N for a given minimum energy consumption and initial state. This information will enhance the usefulness of this and the previous report [4]. Research in this general area is being continued with a detailed study of the saturation problem.

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APPENDICES

I. Derivation of Eq. 16; $\left[H^tDH + F\right] \underline{b} = H^tD\underline{c}$.

Eq. 15 states

$$E = \underline{c}^tD\underline{c} - 2\underline{c}^tDH\underline{b} + \underline{b}^t\left[H^tDH + F\right]\underline{b}$$

$$= \underline{c}^tD\underline{c} - 2\underline{b}^tH^tD\underline{c} + \underline{b}^t\left[H^tDH + F\right]\underline{b}$$

Therefore,

$$\frac{\partial E}{\partial \underline{b}} = \begin{bmatrix} \frac{\partial E}{\partial b_1} \\ \frac{\partial E}{\partial b_2} \\ \cdot \\ \cdot \\ \frac{\partial E}{\partial b_{N-n}} \end{bmatrix} = -2H^tD\underline{c} + 2\left[H^tDH + F\right]\underline{b} \quad .$$

Setting $\frac{\partial E}{\partial \underline{b}} = \underline{0}$, we obtain directly

$$\left[H^tDH + F\right]\underline{b} = H^tD\underline{c} \quad .$$

Since we obtain on further differentation,

$$\frac{\partial^2 E}{\partial \underline{b}^2} = \begin{bmatrix} \frac{\partial^2 E}{\partial b_1^2} \\ \frac{\partial^2 E}{\partial b_2^2} \\ \cdot \\ \cdot \\ \frac{\partial^2 E}{\partial b_{N-n}^2} \end{bmatrix}$$, which has positive elements,

Eq. 15 is the condition for a minimum of the cost function E.

II . Given $B = I + H F^{-1} H^t D$, show that B^{-1} exists.

We shall prove that B is a positive definite matrix, from which it follows directly that the inverse of B does exist. The background material for this Appendix is contained in Hadley[6].

For convenience let $N-n = p$. Let $F^{-1} = F_1 F_1$, where F_1 is the p x p diagonal matrix with elements

$$f^1_{jj} = 1/\sqrt{f_{jj}} \quad , \; j = 1, 2, \ldots p.$$

Then

$$HF_1F_1H^t = \begin{bmatrix} \underline{w}_1^t \\ \underline{w}_2^t \\ \cdot \\ \cdot \\ \underline{w}_n^t \end{bmatrix} \begin{bmatrix} \underline{w}_1 & \underline{w}_2 & \cdots & \underline{w}_n \end{bmatrix}$$

where

$$\underline{w}_i^t = \left(\frac{h_{i1}}{\sqrt{f_{11}}} , \frac{h_{i2}}{\sqrt{f_{22}}} , \ldots \frac{h_{ip}}{\sqrt{f_{pp}}} \right)$$

for $i = 1, 2, \ldots n$. h_{ij} is the i,j-th element of H. We can then express $HF^{-1}H^tD$ in the form,

$$\begin{bmatrix} \underline{w}_1^t \underline{w}_1 & \underline{w}_1^t \underline{w}_2 & \cdots & \underline{w}_1^t \underline{w}_n \\ \underline{w}_2^t \underline{w}_1 & \underline{w}_2^t \underline{w}_2 & \cdots & \underline{w}_2^t \underline{w}_n \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \underline{w}_n^t \underline{w}_1 & \underline{w}_n^t \underline{w}_2 & \cdots & \underline{w}_n^t \underline{w}_n \end{bmatrix} \begin{bmatrix} d_{11} & 0 & \cdot\ \cdot & 0 \\ 0 & d_{22} & \cdot\ \cdot & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \cdots & d_{nn} \end{bmatrix} .$$

Let us call the first matrix W. Then

$$HF^{-1}H^tD = WD$$

W is simply the Grammian of the vectors w_i , $i = 1, 2, \ldots n$, and thus each of the principal minors of W is positive semi-definite, since each principal minor is itself a Grammian. The principal minor of WD of order m is then $d_{11} d_{22} \cdots d_{mm}$ times the principal minor of order m of W. Thus $HF^{-1}H^tD$ is positive semi-definite, and its eigenvalues are $\mu_i \geqslant 0$ for $i = 1, 2, \cdots n$.

The eigenvalues of $I + HF^{-1}H^tD$ are then $1 + \mu_i$, $i = 1, 2, . . n$, which are clearly positive. Thus $B = I + HF^{-1}H^tD$ is itself positive definite, and B^{-1} does exist.

Q.E.D.

REFERENCES

1. Zadeh, L. A., and C. A. Desoer, "Linear System Theory", McGraw-Hill Book Co., Inc., New York, 1963.

2. Kalman, R. E., and J. E. Bertram, "General Synthesis Procedure for Computer Control of Single Loop and Multiloop Linear Systems", Trans. AIEE, Vol. 77, Part II, 1959, pp. 602-609.

3. Kalman, R. E., "Optimum Nonlinear Control of Saturating Systems by Intermittent Action", IRE WESCON Record, Vol. 1, Part 5, 1957, pp. 130-135.

4. Revington, A. M., and J. C. Hung, "Design of Minimum Energy Discrete-Data Control Systems", Scientific Report No. 7, Control Theory Group, Dept. of Elec. Engg., The University of Tennessee, January 1965.

5. Kalman, R. E., "On the General Theory of Control Systems", Proc. First International Congress on Automatic Control, Moscow, Vol. 1, 1961, pp. 481-492.

6. Hadley, G., "Linear Algebra", Addison-Wesley Publishing Co., Inc., Reading, Mass., 1961.

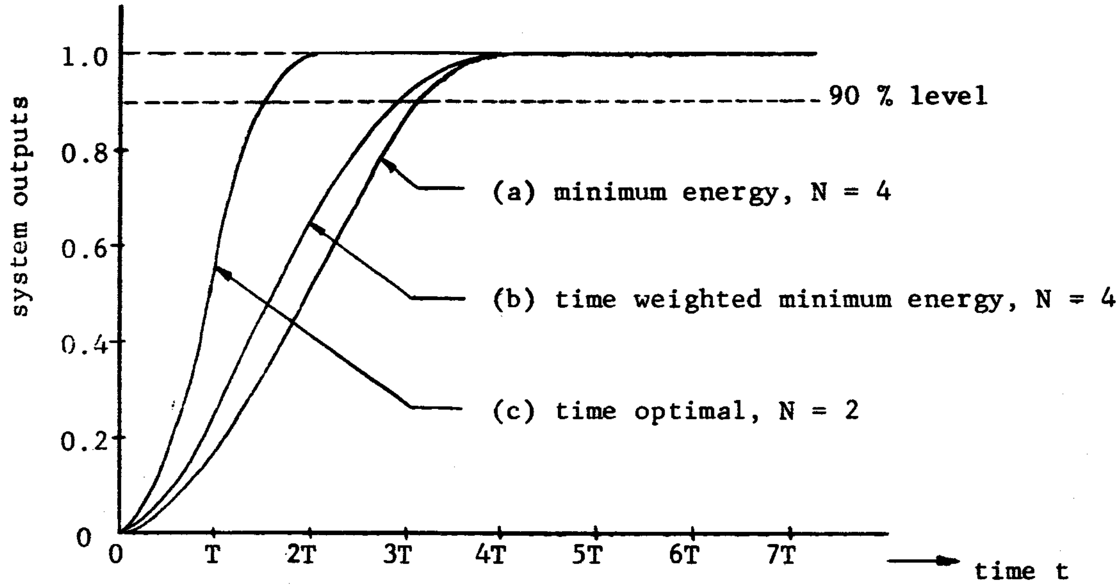

Fig. 1. System output responses

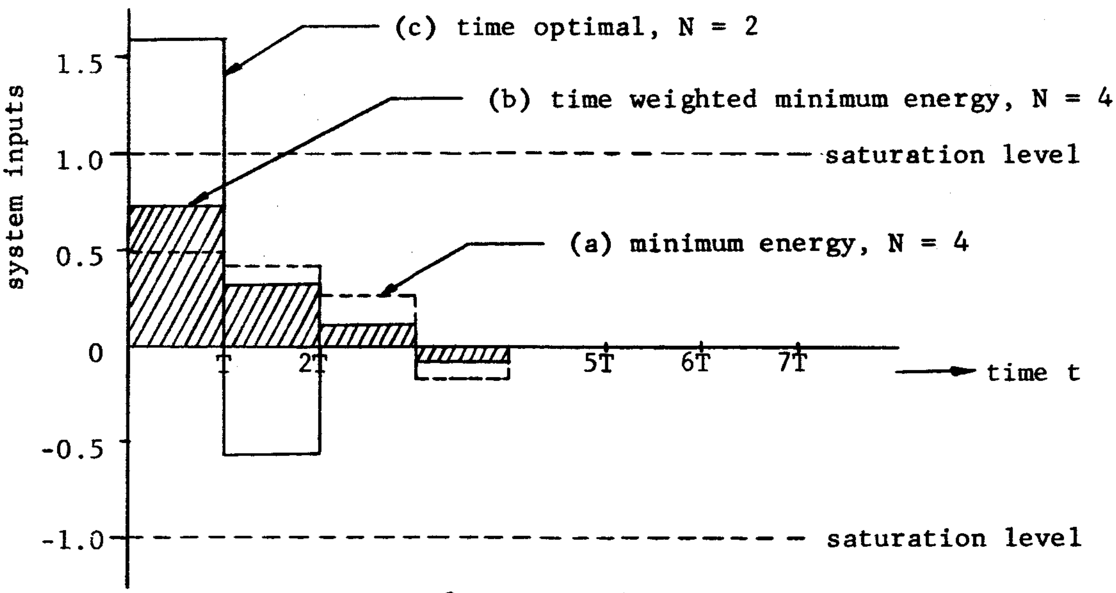

Fig. 2. System inputs